

TRANSPORT AND GENERATION OF MACROSCOPICALLY MODULATED WAVES IN DIATOMIC CHAINS

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ABSTRACT. We derive and justify analytically the dynamics of a small macroscopically modulated amplitude of a single plane wave in a nonlinear diatomic chain with stabilizing on-site potentials including the case where a wave generates another wave via self-interaction. More precisely, we show that in typical chains acoustical waves can generate optical but not acoustical waves, while optical waves are always closed with respect to self-interaction.

1. INTRODUCTION

The present work constitutes a generalization of previous work of the author, see [3], to a case of vector-valued displacement in nonlinear lattices. As the technically most simple but yet generic case we consider a nonlinear diatomic chain. For the physical derivation, interpretation and discussion of several applications of the harmonic diatomic chain we refer to [2]. Various questions concerning diatomic lattices have been addressed up to now, see e.g. [1, 4, 7, 8, 9]. Here we focus on the analytical justification of the dynamics of small macroscopic amplitude modulations, see (12). More precisely, we consider the diatomic chain

$$(1) \quad \begin{cases} \ddot{x}_{2j+1} &= V'_1(x_{2j+2} - x_{2j+1}) - V'_1(x_{2j+1} - x_{2j}) - W'_1(x_{2j+1}), \\ \ddot{x}_{2j} &= V'_2(x_{2j+1} - x_{2j}) - V'_2(x_{2j} - x_{2j-1}) - W'_2(x_{2j}), \end{cases} \quad j \in \mathbb{Z},$$

with nearest-neighbor interaction and on-site potentials $V_i, W_i \in C^4(\mathbb{R})$, $i = 1, 2$, such that

$$(2) \quad \begin{cases} V'_i(x) &= v_{i,1}x + v_{i,2}x^2 + \tilde{V}'_i(x), \quad \tilde{V}'_i(x) = O(|x|^3), \\ W'_i(x) &= w_{i,1}x + w_{i,2}x^2 + \tilde{W}'_i(x), \quad \tilde{W}'_i(x) = O(|x|^3). \end{cases}$$

Setting $u_j = \begin{pmatrix} u_{j,1} \\ u_{j,2} \end{pmatrix} := \begin{pmatrix} x_{2j+1} \\ x_{2j} \end{pmatrix}$, $j \in \mathbb{Z}$, and using the Taylor-expansions (2), the diatomic chain (1) takes the form

$$(3) \quad \begin{aligned} \ddot{u} &= \mathcal{L}u + \mathcal{M}(u), \\ (\mathcal{L}u)_j &:= \begin{pmatrix} v_{1,1}(u_{j+1,2} - 2u_{j,1} + u_{j,2}) - w_{1,1}u_{j,1} \\ v_{2,1}(u_{j,1} - 2u_{j,2} - u_{j-1,1}) - w_{2,1}u_{j,2} \end{pmatrix}, \\ (\mathcal{M}(u))_j &:= \begin{pmatrix} v_{1,2}((u_{j+1,2} - u_{j,1})^2 - (u_{j,1} - u_{j,2})^2) - w_{1,2}u_{j,1}^2 \\ v_{2,2}((u_{j,1} - u_{j,2})^2 - (u_{j,2} - u_{j-1,1})^2) - w_{2,2}u_{j,2}^2 \end{pmatrix} + \end{aligned}$$

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$$+ \left(\tilde{V}'_1(u_{j+1,2} - u_{j,1}) - \tilde{V}'_1(u_{j,1} - u_{j,2}) - \tilde{W}'_1(u_{j,1}) \right) \\ + \left(\tilde{V}'_2(u_{j,1} - u_{j,2}) - \tilde{V}'_2(u_{j,2} - u_{j-1,1}) - \tilde{W}'_2(u_{j,2}) \right).$$

The linearized model $\ddot{u} = \mathcal{L}u$ admits for non-trivial plane-wave solutions

$$u = A\mathbf{E} + \text{c.c.}, \quad \mathbf{E}(t, j) := e^{i(\omega t + j\vartheta)}, \quad A := \begin{pmatrix} A^{(1)} \\ A^{(2)} \end{pmatrix} \in \mathbb{C}^2,$$

provided the frequency $\omega \in \mathbb{R}$ and the wave number $\vartheta \in (-\pi, \pi]$ satisfy the dispersion relation

$$(4) \quad \det H(\omega, \vartheta) = 0, \quad H(\omega, \vartheta) := \begin{pmatrix} \omega^2 - c_1 & v_{1,1}(e^{i\vartheta} + 1) \\ v_{2,1}(1 + e^{-i\vartheta}) & \omega^2 - c_2 \end{pmatrix},$$

where $c_i := 2v_{i,1} + w_{i,1}$. This is equivalent to

$$(5) \quad \omega^2 = \omega_{\pm}^2(\vartheta) := \frac{c_1 + c_2}{2} \pm \frac{1}{2} \sqrt{(c_1 - c_2)^2 + 8v_{1,1}v_{2,1}(\cos \vartheta + 1)}.$$

Assuming $c_1 + c_2 > 0$, $c_1 c_2 > 4v_{1,1}v_{2,1} > 0$, we obtain

$$(6) \quad \omega_{\pm}(\vartheta) := \sqrt{\frac{1}{2} \left(c_1 + c_2 \pm \sqrt{(c_1 - c_2)^2 + 8v_{1,1}v_{2,1}(\cos \vartheta + 1)} \right)} > 0$$

for all $\vartheta \in (-\pi, \pi]$ and the additional assumption $c_1 \neq c_2$ yields the strict separation of the optical and acoustical branches of the frequency,

$$2\omega_+^2(\vartheta) \geq c_1 + c_2 + |c_1 - c_2| > c_1 + c_2 - |c_1 - c_2| \geq 2\omega_-^2(\vartheta) \quad \forall \vartheta \in (-\pi, \pi].$$

All of the above assumptions are satisfied in the case $w_{i,1} > 0$, $4v_{i,1} + w_{i,1} > 0$, $v_{1,1}v_{2,1} > 0$, $2v_{2,1} + w_{2,1} > 2v_{1,1} + w_{1,1}$, which we assume in the following.

The eigenvectors A to the eigenfrequencies $\omega = \omega_{\pm}(\vartheta)$ are given by

$$(7) \quad A^{(2)} = -\rho A^{(1)}, \quad \rho := \frac{\omega^2 - c_1}{v_{1,1}(e^{i\vartheta} + 1)} = \frac{v_{2,1}(e^{-i\vartheta} + 1)}{\omega^2 - c_2} \neq 0, \quad \text{if } \vartheta \neq \pm\pi$$

and

$$(8) \quad A = \begin{pmatrix} A^{(1)} \\ 0 \end{pmatrix} \quad \text{for } \omega = \omega_-(\pm\pi), \quad A = \begin{pmatrix} 0 \\ A^{(2)} \end{pmatrix} \quad \text{for } \omega = \omega_+(\pm\pi).$$

The plan of the paper is as follows. In Section 2 we discuss whether a given plane wave solution \mathbf{E} can generate via self-interaction another plane wave \mathbf{E}^2 . Then, taking into account also this possibility, in Section 3 we derive formally the macroscopic equations for the first order amplitudes $A_{1,n}$ of two waves $n = 1, 2$, and finally, in Section 4, we justify the derived equations.

2. RESONANCES

Since we are interested in the self-interaction of a plane wave \mathbf{E} , which means that \mathbf{E}^2 is also a plane wave, in a diatomic chain we are interested in resonance conditions like the ones on the left hand side below. Making in (6) the substitutions $c := (\cos \vartheta + 1)/2 \in [0, 1]$, $d_1 := (c_1 + c_2)^2/f > 0$, $d_2 := (c_1 - c_2)^2/f > 0$ with $f := 16v_{1,1}v_{2,1} > 0$ and $d_1 - d_2 > 1$, the problem of finding a $\vartheta \in (-\pi, \pi]$ satisfying one of these resonance conditions is equivalent to finding a $c \in [0, 1]$ for given $d_1 > d_2 + 1 > 1$ satisfying the corresponding equation on the right hand side:

$$(9) \quad \begin{aligned} 2\omega_{(\pm)}(\vartheta) = \omega_{\pm}(2\vartheta) &\Leftrightarrow 4 \left(\sqrt{d_1}(\pm) \sqrt{d_2 + c} \right) = \sqrt{d_1} \pm \sqrt{d_2 + (2c - 1)^2} \\ &\Leftrightarrow 3\sqrt{d_1} = (\mp) 4\sqrt{d_2 + c} \pm \sqrt{d_2 + (2c - 1)^2}. \end{aligned}$$

By the positivity of all appearing square roots we immediately see that a resonance $2\omega_+(\vartheta) = \omega_-(2\vartheta)$, i.e., an optical wave generating an acoustical one, is not possible. Moreover, since

$$3\sqrt{d_1} > \sqrt{d_1} > \sqrt{d_2 + 1} > -4\sqrt{d_2 + c} + \sqrt{d_2 + (2c - 1)^2},$$

we see that an optical wave can not generate another optical one, i.e., $2\omega_+(\vartheta) \neq \omega_+(2\vartheta) \forall \vartheta \in (-\pi, \pi]$. Thus, *an optical wave is closed under self-interaction of order 2*.

However, *an acoustical wave can generate an optical one by self-interaction*, i.e., for appropriate choice of the harmonic parts of the interaction and on-site potentials there exist $\vartheta \in (-\pi, \pi]$ such that $2\omega_-(\vartheta) = \omega_+(2\vartheta)$. After taking squares on the left and right hand sides, the corresponding condition (9) reads

$$(10) \quad 9d_1 = 17d_2 + 16c + (2c - 1)^2 + 8\sqrt{d_2 + c}\sqrt{d_2 + (2c - 1)^2},$$

and we want to prove the existence of a $c \in [0, 1]$ that satisfies this condition for the d_1, d_2 given above. We restrict ourselves to the case $v_{1,1} = a > 0$, $v_{2,1} = \gamma a$, $\gamma > 1$, $w_{1,1} = w_{2,1} = b > 0$. This setting satisfies all conditions posed so far on the harmonic coefficients, and we obtain

$$(11) \quad d_1 = \frac{(\gamma + 1)^2}{4\gamma} + \frac{1}{\gamma} \left((\gamma + 1) \frac{b}{a} + \frac{b^2}{a^2} \right), \quad d_2 = \frac{(\gamma - 1)^2}{4\gamma} =: \delta$$

(which obviously satisfies $d_1 > d_2 + 1 > 1$). Inserting these values into (10), we get

$$\frac{9}{\gamma} \left((\gamma + 1) \frac{b}{a} + \frac{b^2}{a^2} \right) = 8\delta - 9 + 16c + (2c - 1)^2 + 8\sqrt{\delta + c}\sqrt{\delta + (2c - 1)^2}.$$

Hence, for every $c \in [0, 1]$ such that $16c \geq 9 - 8\delta$ there exists a $\frac{b}{a}$ such that (10) is satisfied. Since $\delta > 0$, we can always find such a c .

Furthermore, the resonance condition for the generation of an acoustical wave from an acoustical one, $2\omega_-(\vartheta) = \omega_-(2\vartheta)$, is equivalent to

$$3\sqrt{d_1} = 4\sqrt{d_2 + c} - \sqrt{d_2 + (2c - 1)^2}$$

Concerning the case just considered, we observe that for $d_2 = \delta$, the r.h.s. is nonnegative only for $c \in [c_e, 1]$ with $c_e := \max\{0, \frac{5 - \sqrt{15\delta + 24}}{2}\}$ (and hence for all $c \in [0, 1]$ when $\delta \geq 1/15$). Restricting our analysis to the set $[c_e, 1]$ (non-empty for all $\delta > 0$), we obtain by squaring and insertion of the values (11) as above

$$\frac{9}{\gamma} \left((\gamma + 1) \frac{b}{a} + \frac{b^2}{a^2} \right) = 8\delta - 9 + 16c + (2c - 1)^2 - 8\sqrt{\delta + c}\sqrt{\delta + (2c - 1)^2},$$

although with a minus sign in front of the square root. Due to the existent on-site potential (where $b > 0$), in order to obtain resonances the r.h.s. g needs to be strictly positive for some $c \in [c_e, 1]$. However, a careful analysis reveals that $g(c) \leq 0$ for $c \in [c_e, 1]$, and we obtain that in the case $v_{2,1} = \gamma a > a = v_{1,1}$, $w_{1,1} = w_{2,1} = b > 0$, *an acoustical wave can not generate another acoustical one by self-interaction*.

Finally, we conclude by showing that $\omega_-(\vartheta) + \omega_+(2\vartheta) \neq \omega_+(3\vartheta)$ for all $\vartheta \in (-\pi, \pi]$. Indeed, after squaring the left and right hand sides we see that the equality is equivalent to

$$\begin{aligned} & -\sqrt{d_2 + c} + \sqrt{d_2 + (2c - 1)^2} + 2\sqrt{(\sqrt{d_1} - \sqrt{d_2 + c})(\sqrt{d_1} + \sqrt{d_2 + (2c - 1)^2})} \\ & = \sqrt{d_2 + (4c - 3)^2} - \sqrt{d_1} \end{aligned}$$

for $d_1 > d_2 + 1 > 1$. Since $(4c - 3)^2 c = (\cos(3\vartheta) + 1)/2 \in [0, 1]$, the r.h.s. of this equation is always < 0 , and it suffices to show that the l.h.s. is ≥ 0 even for $c \geq (2c - 1)^2$. Hence, since $d_1 > d_2 + 1$, it is sufficient to show that the l.h.s. with $\sqrt{d_1}$ replaced by $\sqrt{d_2 + 1}$ is ≥ 0 for $c \in [1/4, 1]$. Comparing in this modified l.h.s. the square of the first two terms with the square of the third one, and adding a suitable term, this is equivalent to showing that

$$4(1 - c) \geq (\sqrt{d_2 + c} - \sqrt{d_2 + (2c - 1)^2}) \left((\sqrt{d_2 + c} - \sqrt{d_2 + (2c - 1)^2}) + 4(\sqrt{d_2 + 1} - \sqrt{d_2 + c}) \right)$$

for $c \in [1/4, 1]$. Since the r.h.s. is positive and strictly decreasing as a function of $d_2 > 0$ when $c > (2c - 1)^2$, it suffices to show

$$g(c) := 4(1 - c) - (\sqrt{c} - \sqrt{(2c - 1)^2}) \left((\sqrt{c} - \sqrt{(2c - 1)^2}) + 4(1 - \sqrt{c}) \right) \geq 0$$

for $c \in [1/4, 1]$, which holds true (as an elementary analysis shows), with $g(1) = 0$.

3. FORMAL DERIVATION

We are interested in solutions of (3) which in first order in ε are a sum of two macroscopically modulated plane-wave solutions with small amplitudes

$$(12) \quad u = U_\varepsilon^{A,1} + O(\varepsilon^2), \quad (U_\varepsilon^{A,1})_j(t) := \varepsilon \sum_{n=1}^2 A_{1,n}(\varepsilon t, \varepsilon j) \mathbf{E}_n(t, j) + \text{c.c.}$$

where $A_{1,n} = (A_{1,n}^{(1)}, A_{1,n}^{(2)})^T : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}^2$ and $\mathbf{E}_n(t, j) := e^{i(\omega_n t + j \vartheta_n)}$ with (ω_n, ϑ_n) satisfying (4).

However, due to the scaling of $A_{1,n}$ by ε and the macroscopic nature of its time and space variables, its dynamics will include terms of second order in ε . Hence, taking into account the nonlinearity of our original system (3) and the fact that we consider two different plane waves, we insert into (3) the *improved approximation*

$$(13) \quad U_\varepsilon^{A,2} := U_\varepsilon^{A,1} + \varepsilon^2 \left(\sum_{n=1}^2 (A_{2,n} \mathbf{E}_n + A_{2,(n,n)} \mathbf{E}_n^2) + A_{2,(1,2)} \mathbf{E}_1 \mathbf{E}_2 + A_{2,(1,-2)} \mathbf{E}_1 \mathbf{E}_{-2} + \frac{1}{2} A_{2,(1,-1)} + \text{c.c.} \right),$$

where $A_{2,\iota} = (A_{2,\iota}^{(1)}, A_{2,\iota}^{(2)})^T : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}^2$, $\iota \in \{1, 2\} \cup I$, $I := \{(1, 1), (2, 2), (1, 2), (1, -2), (1, -1)\}$, are again functions of the macroscopic variables $\tau = \varepsilon t$, $y = \varepsilon j$, and where $\mathbf{E}_{-n} = \overline{\mathbf{E}_n}$. Thereby, we use the Taylor expansions

$$A_{1,n}^{(i)}(\cdot, \cdot \pm \varepsilon) = A_{1,n}^{(i)} \pm \varepsilon \partial_y A_{1,n}^{(i)} + \varepsilon^2 \frac{1}{2} \partial_y^2 A_{1,n,\xi_1 \pm}^{(i)}, \quad \partial_y^2 A_{1,n,\xi_1 \pm}^{(i)} := \partial_y^2 A_{1,n}^{(i)}(\tau, y \pm \xi_1 \varepsilon),$$

$$A_{2,\iota}^{(i)}(\cdot, \cdot \pm \varepsilon) = A_{2,\iota}^{(i)} \pm \varepsilon \partial_y A_{2,\iota,\xi_2 \pm}^{(i)}, \quad \partial_y A_{2,\iota,\xi_2 \pm}^{(i)} := \partial_y A_{2,\iota}^{(i)}(\tau, y \pm \xi_2 \varepsilon)$$

with $\xi_1, \xi_2 \in (0, 1)$, assuming $A_{1,n}(\tau, \cdot) \in C^2(\mathbb{R}; \mathbb{C}^2)$, $A_{2,\iota}(\tau, \cdot) \in C^1(\mathbb{R}; \mathbb{C}^2)$.

Carrying out the usual (lengthy but straightforward) formal expansion in terms of ε and \mathbf{E}_n , we obtain that $\ddot{U}_\varepsilon^{A,2} = \mathcal{L} U_\varepsilon^{A,2} + \mathcal{M}(U_\varepsilon^{A,2})$ is equivalent to

$$\varepsilon \left\{ \sum_{n=1}^2 H(\omega_n, \vartheta_n) A_{1,n} \mathbf{E}_n + \text{c.c.} \right\} +$$

$$\begin{aligned}
& + \varepsilon^2 \left\{ \sum_{n=1}^2 \left(\begin{pmatrix} -2i\omega_n \partial_\tau A_{1,n}^{(1)} + v_{1,1} e^{i\vartheta_n} \partial_y A_{1,n}^{(2)} \\ -2i\omega_n \partial_\tau A_{1,n}^{(2)} - v_{2,1} e^{-i\vartheta_n} \partial_y A_{1,n}^{(1)} \end{pmatrix} + H(\omega_n, \vartheta_n) A_{2,n} \right) \mathbf{E}_n \right. \\
& + \sum_{n=1}^2 \left(H(2\omega_n, 2\vartheta_n) A_{2,(n,n)} + K_{(n,n)} \right) \mathbf{E}_n^2 \\
& + \left(H(\omega_1 + \omega_2, \vartheta_1 + \vartheta_2) A_{2,(1,2)} + K_{(1,2)} \right) \mathbf{E}_1 \mathbf{E}_2 \\
& + \left(H(\omega_1 - \omega_2, \vartheta_1 - \vartheta_2) A_{2,(1,-2)} + K_{(1,-2)} \right) \mathbf{E}_1 \mathbf{E}_{-2} \\
& \left. + \frac{1}{2} H(0, 0) A_{2,(1,-1)} + K_{(1,-1)} + \text{c.c.} \right\} + \text{res}(U_\varepsilon^{A,2}) = 0
\end{aligned}$$

with the explicit expressions for K_ι , $\iota \in I$, and $\text{res}(U_\varepsilon^{A,2}) = \mathcal{O}(\varepsilon^3)$ given in the Appendix. Hence, in order for our ansatz (13) to satisfy (3) up to order ε , taking into account that $\mathbf{E}_1 \neq \mathbf{E}_2$, the systems $H(\omega_n, \vartheta_n) A_{1,n} = 0$ have to be satisfied. As we have already seen, since $\det H(\omega_n, \vartheta_n) = 0$, this gives the relation between first and second component of $A_{1,n}$ (7), (8) with $A, \rho, \omega, \vartheta$ replaced by $A_{1,n}, \rho_n, \omega_n, \vartheta_n$.

Next, we assume that

$$(14) \quad \det H(\omega, \vartheta) \neq 0 \quad \text{for} \quad (\omega, \vartheta) = (2\omega_n, 2\vartheta_n), (\omega_1 \pm \omega_2, \vartheta_1 \pm \vartheta_2),$$

which means in particular that $\mathbf{E}_n^2, \mathbf{E}_1 \mathbf{E}_2, \mathbf{E}_1 \mathbf{E}_{-2} \neq \mathbf{E}_1, \mathbf{E}_2$. (Note here that $\det H(0, 0) \neq 0$ is always satisfied due to our stability assumption $c_1 c_2 > 4v_{1,1} v_{2,1}$.) In this case and for $\vartheta_n \neq \pm\pi$, we obtain from the equations for $\varepsilon^2 \mathbf{E}_n$

$$\begin{aligned}
(15) \quad \rho_n A_{2,n}^{(1)} + A_{2,n}^{(2)} &= \frac{1}{v_{1,1}(e^{i\vartheta_n} + 1)} (2i\omega_n \partial_\tau A_{1,n}^{(1)} - v_{1,1} e^{i\vartheta_n} \partial_y A_{1,n}^{(2)}) \\
&= \frac{1}{\omega_n^2 - c_2} (2i\omega_n \partial_\tau A_{1,n}^{(2)} + v_{2,1} e^{-i\vartheta_n} \partial_y A_{1,n}^{(1)}).
\end{aligned}$$

Inserting $A_{1,n}^{(2)} = -\rho_n A_{1,n}^{(1)}$, and noting that (5) gives

$$\omega'_\pm(\vartheta) = \frac{-v_{1,1} v_{2,1} \sin \vartheta}{\omega_\pm(\vartheta)(2\omega_\pm^2(\vartheta) - c_1 - c_2)},$$

we obtain from the equality of the right hand sides of (15)

$$(16) \quad \partial_\tau A_{1,n}^{(1)} - \omega'_\pm(\vartheta_n) \partial_y A_{1,n}^{(1)} = 0 \quad \text{for} \quad \omega_n = \omega_\pm(\vartheta_n).$$

Analogously, in the case $\vartheta_n = \pm\pi$ we get from (8) (for $A = A_{1,n}$)

$$(17) \quad \partial_\tau A_{1,n}^{(1)} = 0, \quad A_{2,n}^{(2)} = \frac{v_{2,1}}{c_2 - c_1} \partial_y A_{1,n}^{(1)} \quad \text{for} \quad \omega_n^2 = \omega_-^2(\pm\pi) = c_1,$$

$$(18) \quad \partial_\tau A_{1,n}^{(2)} = 0, \quad A_{2,n}^{(1)} = \frac{v_{1,1}}{c_2 - c_1} \partial_y A_{1,n}^{(2)} \quad \text{for} \quad \omega_n^2 = \omega_+^2(\pm\pi) = c_2.$$

Thus, we conclude that if the non-resonance conditions (14) hold, which means in particular that neither wave generates a new one via self-interaction, the dynamics of the amplitudes $A_{1,n}$ are given by uncoupled transport equations where the velocity is the group velocity of the corresponding carrier wave. Hence, setting in particular $A_{1,2}(0, \cdot) = 0$ we obtain that the dynamics of $A_{1,1}$ are given, unsurprisingly, by a homogeneous transport equation. Moreover, since the K_ι , $\iota \in I$, are known, as they depend only on the first order amplitudes $A_{1,n}$ (see Appendix), and since (15), (17)₂, (18)₂ determine the relation between the components of $A_{2,n}$, we obtain by (14) all $A_{2,\iota}$ except for one component of $A_{2,n}$, which can be assumed to be equivalently vanishing.

However, it is possible that $(\omega_2, \vartheta_2) = (2\omega_1, 2\vartheta_1)$, i.e. $\mathbf{E}_2 = \mathbf{E}_1^2$, namely for $\omega_1 = \omega_-(\vartheta_1)$, $\omega_2 = \omega_+(\vartheta_2)$, which moreover implies that $\mathbf{E}_1^3 = \mathbf{E}_1\mathbf{E}_2$, $\mathbf{E}_1^4 = \mathbf{E}_2^2$ do not characterize plane waves, as we have shown in Section 2. In this case the formal expansion gives

$$\begin{aligned} & \varepsilon \left\{ \sum_{n=1}^2 H(\omega_n, \vartheta_n) A_{1,n} \mathbf{E}_n + \text{c.c.} \right\} \\ & + \varepsilon^2 \left\{ \left(\begin{pmatrix} -2i\omega_1 \partial_\tau A_{1,1}^{(1)} + v_{1,1} e^{i\vartheta_1} \partial_y A_{1,1}^{(2)} \\ -2i\omega_1 \partial_\tau A_{1,1}^{(2)} - v_{2,1} e^{-i\vartheta_1} \partial_y A_{1,1}^{(1)} \end{pmatrix} + H(\omega_1, \vartheta_1) A_{2,1} + \bar{K}_{(1,-2)} \right) \mathbf{E}_1 \right. \\ & \quad + \left(\begin{pmatrix} -2i\omega_2 \partial_\tau A_{1,2}^{(1)} + v_{1,1} e^{i\vartheta_2} \partial_y A_{1,2}^{(2)} \\ -2i\omega_2 \partial_\tau A_{1,2}^{(2)} - v_{2,1} e^{-i\vartheta_2} \partial_y A_{1,2}^{(1)} \end{pmatrix} + H(\omega_2, \vartheta_2) A_{2,2} + K_{(1,1)} \right) \mathbf{E}_2 \\ & \quad + (H(2\omega_2, 2\vartheta_2) A_{2,(2,2)} + K_{(2,2)}) \mathbf{E}_1^4 + (H(\omega_1 + \omega_2, \vartheta_1 + \vartheta_2) A_{2,(1,2)} + K_{(1,2)}) \mathbf{E}_1^3 \\ & \quad \left. + \frac{1}{2} H(0, 0) A_{2,(1,-1)} + K_{(1,-1)} + \text{c.c.} \right\} + \text{res}(U_\varepsilon^{A,2}) = 0 \end{aligned}$$

The equations for $\varepsilon \mathbf{E}_n$ are the same as before, and hence (7) and (8) (with $A_{1,n}$, ρ_n , ω_n , ϑ_n) are still valid. Then, using ρ_n , we obtain from the equations for $\varepsilon^2 \mathbf{E}_n$ in the case $\vartheta_n \neq \pm\pi$

$$\begin{aligned} \rho_1 A_{2,1}^{(1)} + A_{2,1}^{(2)} &= \frac{1}{v_{1,1}(e^{i\vartheta_1} + 1)} \left(2i\omega_1 \partial_\tau A_{1,1}^{(1)} - v_{1,1} e^{i\vartheta_1} \partial_y A_{1,1}^{(2)} + \bar{K}_{(1,-2)}^{(1)} \right) \\ &= \frac{1}{\omega_1^2 - c_2} \left(2i\omega_1 \partial_\tau A_{1,1}^{(2)} + v_{2,1} e^{-i\vartheta_1} \partial_y A_{1,1}^{(1)} + \bar{K}_{(1,-2)}^{(2)} \right), \\ \rho_2 A_{2,2}^{(1)} + A_{2,2}^{(2)} &= \frac{1}{v_{1,1}(e^{i\vartheta_2} + 1)} \left(2i\omega_2 \partial_\tau A_{1,2}^{(1)} - v_{1,1} e^{i\vartheta_2} \partial_y A_{1,2}^{(2)} + K_{(1,1)}^{(1)} \right) \\ &= \frac{1}{\omega_2^2 - c_2} \left(2i\omega_2 \partial_\tau A_{1,2}^{(2)} + v_{2,1} e^{-i\vartheta_2} \partial_y A_{1,2}^{(1)} + K_{(1,1)}^{(2)} \right), \end{aligned}$$

and inserting (7) into the equalities on the right hand side we get for $\vartheta_1 \neq \pm\frac{\pi}{2}, \pm\pi$, $\vartheta_2 = 2\vartheta_1$, $\omega_1 = \omega_-(\vartheta_1)$, $\omega_2 = \omega_+(\vartheta_2) = 2\omega_1$

$$(19) \quad \begin{cases} \partial_\tau A_{1,1}^{(1)} - \omega'_-(\vartheta_1) \partial_y A_{1,1}^{(1)} &= \frac{d_1}{i\omega_1} \frac{\omega_1^2 - c_2}{(\omega_1^2 - c_1) + (\omega_1^2 - c_2)} \bar{A}_{1,1}^{(1)} A_{1,2}^{(1)}, \\ \partial_\tau A_{1,2}^{(1)} - \omega'_+(\vartheta_2) \partial_y A_{1,2}^{(1)} &= \frac{d_2}{2i\omega_2} \frac{\omega_2^2 - c_2}{(\omega_2^2 - c_1) + (\omega_2^2 - c_2)} (A_{1,1}^{(1)})^2 \end{cases}$$

with

$$\begin{aligned} d_1 &:= dv_{2,2} \left(\frac{e^{-i\vartheta_1} - 1}{\bar{\rho}_1 \rho_2} + \frac{e^{-i2\vartheta_1} - 1}{\rho_2} + \frac{e^{i\vartheta_1} - 1}{\bar{\rho}_1} \right) \\ &\quad + v_{1,2} \left(\bar{\rho}_1 \rho_2 (e^{i\vartheta_1} - 1) + \rho_2 (e^{i2\vartheta_1} - 1) + \bar{\rho}_1 (e^{-i\vartheta_1} - 1) \right) + dw_{2,2} - w_{1,2}, \\ d_2 &:= dv_{2,2} \left(\frac{e^{-i2\vartheta_1} - 1}{\rho_1^2} + 2 \frac{e^{-i\vartheta_1} - 1}{\rho_1} \right) + v_{1,2} \left(\rho_1^2 (e^{i2\vartheta_1} - 1) + 2\rho_1 (e^{i\vartheta_1} - 1) \right) \\ &\quad + dw_{2,2} - w_{1,2}, \\ d &:= \frac{v_{1,1} v_{2,1}^2 (2 + 4 \cos \vartheta_1 + 2 \cos \vartheta_2)}{(\omega_1^2 - c_2)^2 (\omega_2^2 - c_2)}. \end{aligned}$$

Analogously, for $\vartheta_1 = \pm \frac{\pi}{2}$, $\vartheta_2 = 2\vartheta_1$, $\omega_1 = \omega_-(\vartheta_1)$, $\omega_2 = \omega_+(\vartheta_2) = \sqrt{c_2} = 2\omega_1$ we obtain

$$(20) \quad \begin{cases} \partial_\tau A_{1,1}^{(1)} - \omega'_-(\pm \frac{\pi}{2}) \partial_y A_{1,1}^{(1)} &= \frac{d_1}{i\omega_1} \frac{\omega_1^2 - c_2}{\omega_1^2 - c_2 + \omega_1^2 - c_1} \bar{A}_{1,1}^{(1)} A_{1,2}^{(2)} \\ \partial_\tau A_{1,2}^{(2)} &= \frac{1}{2i\omega_2} \left(2v_{2,2}(1 + \rho_1(1 \pm i)) - w_{2,2}\rho_1^2 \right) (A_{1,1}^{(1)})^2 \end{cases}$$

with $d_1 = \left(\frac{v_{2,2}}{v_{2,1}} \mp i \frac{w_{2,2}}{\omega_1^2 - c_2} \right) (\omega_1^2 - c_1) + v_{1,2}(\rho_1(1 \pm i) + 2)$ and

$$\begin{aligned} \rho_1 A_{2,1}^{(1)} + A_{2,1}^{(2)} &= \rho_1 \left(\frac{1 \mp i}{2} \partial_y A_{1,1}^{(1)} - \frac{2i\omega_1}{\omega_1^2 - c_2} \partial_\tau A_{1,1}^{(1)} \right) + 2\rho_1 \left(\frac{v_{2,2}}{v_{2,1}} \mp i \frac{w_{2,2}}{\omega_1^2 - c_2} \right) \bar{A}_{1,1}^{(1)} A_{1,2}^{(2)} \\ A_{2,2}^{(1)} &= \frac{1}{c_2 - c_1} \left(v_{1,1} \partial_y A_{1,2}^{(2)} + \left(2v_{1,2}(\rho_1^2 + \rho_1(1 \mp i)) + w_{1,2} \right) (A_{1,1}^{(1)})^2 \right) \end{aligned}$$

and for $\vartheta_1 = \pm \pi$, $\vartheta_2 = 2\vartheta_1$, $\omega_1 = \omega_-(\vartheta_1) = \sqrt{c_1}$, $\omega_2 = \omega_+(\vartheta_2) = \omega_+(0) = 2\omega_1$ we get, using (8) and (7), the equations (19) with $\omega'_-(\vartheta_1) = \omega'_+(\vartheta_2) = 0$ and $d_1 = d_2 = -w_{1,2}$, and

$$\begin{aligned} A_{2,1}^{(2)} &= \frac{1}{c_2 - c_1} \left(v_{2,1} \partial_y A_{1,1}^{(1)} + 4v_{2,2}(1 + \rho_2) \bar{A}_{1,1}^{(1)} A_{1,2}^{(1)} \right), \\ \rho_2 A_{2,2}^{(1)} + A_{2,2}^{(2)} &= \rho_2 \left(\frac{1}{2} \partial_y A_{1,2}^{(1)} - \frac{2i\omega_2}{\omega_2^2 - c_2} \partial_\tau A_{1,2}^{(1)} \right). \end{aligned}$$

Hence, in the case $\mathbf{E}_2 = \mathbf{E}_1^2$ we obtain two coupled equations for $A_{1,n}^{(1[2, \text{resp.}])}$, the solutions of which (cf. about their well-posedness Lemma 4.2) determine again $U_\varepsilon^{A,2}$ up to one component of $A_{2,n}$. We would like to stress that in order to obtain non-trivial dynamics for $A_{1,1}$ we have to consider also the dynamics of the generated wave $A_{1,2}$ while if only interested in $A_{1,2}$ we could ignore the generating wave $A_{1,1}$, see e.g. (19). Note in this context, that even for initial data $A_{1,2}(0, \cdot) = 0$ an amplitude $A_{1,2} \neq 0$ emerges, which motivates the notion of *generation* of waves.

4. JUSTIFICATION

The equations obtained by the formal derivation constitute only necessary conditions on the amplitudes $A_{1,n}$ of the *ansatz* (12). The purpose of the justification is to show that indeed solutions u of such a form exist.

Theorem 4.1. *Let $V_i, W_i \in C^4(\mathbb{R})$, $i = 1, 2$, in (2) satisfy*

$$(21) \quad v_{1,1} = \frac{v_1}{M}, \quad v_{2,1} = \frac{v_1}{m}, \quad w_{i,1} > 0, \quad 4v_1 + \min\{Mw_{1,1}, mw_{2,1}\} > 0, \quad M, m > 0,$$

let $\omega_2 > \omega_1 > 0$ and $\vartheta_n \in (-\pi, \pi]$, $n = 1, 2$, satisfy $\det H(\omega_n, \vartheta_n) = 0$ and

$$\text{either } \det H(2\omega_n, 2\vartheta_n) \neq 0, \quad \det H(\omega_1 \pm \omega_2, \vartheta_1 \pm \vartheta_2) \neq 0,$$

$$\text{or } (\omega_2, \vartheta_2) = (2\omega_1, 2\vartheta_1 \bmod 2\pi), \quad \det H(k\omega_1, k\vartheta_1) \neq 0, \quad k = 3, 4,$$

with the dispersion matrix H in (4), and let $A_{1,n}^{(1[2])} : [0, \tau_0] \times \mathbb{R} \rightarrow \mathbb{C}$, $\tau_0 > 0$, be, respectively, the unique solutions of either (16) (or (17) or (18)) or (19) (or (20)) with $A_{1,n}^{(1[2])}(0, \cdot) \in H^4(\mathbb{R}; \mathbb{C})$.

Then, for the corresponding approximation $U_\varepsilon^{A,1}$ and every $c > 0$, $\beta \in (1, 3/2]$ there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, \tau_0/\varepsilon]$ any solution u of (3) satisfies

$$\left\| \begin{pmatrix} u - U_\varepsilon^{A,1} \\ \dot{u} - \dot{U}_\varepsilon^{A,1} \end{pmatrix} (0) \right\|_{(\ell^2)^4} \leq c\varepsilon^\beta \quad \Rightarrow \quad \left\| \begin{pmatrix} u - U_\varepsilon^{A,1} \\ \dot{u} - \dot{U}_\varepsilon^{A,1} \end{pmatrix} (t) \right\|_{(\ell^2)^4} \leq C\varepsilon^\beta.$$

Proof. The idea of the proof is classical, see e.g. [5]. We write the microscopic model (3) as a first order system in $Y := (\ell^2)^4$ with $(\ell^2)^4 = (\ell^2)^2 \times (\ell^2)^2 = (\ell^2 \times \ell^2) \times (\ell^2 \times \ell^2)$ and $\ell^2 = \ell^2(\mathbb{Z})$,

$$(22) \quad \dot{\tilde{u}} = \tilde{\mathcal{L}}\tilde{u} + \tilde{\mathcal{M}}(\tilde{u}) \quad \text{with} \quad \tilde{u} := \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad \tilde{\mathcal{L}} := \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{L} & 0 \end{pmatrix}, \quad \tilde{\mathcal{M}}(\tilde{u}) := \begin{pmatrix} 0 \\ \mathcal{M}(u) \end{pmatrix},$$

where $\mathcal{I} : (\ell^2)^2 \rightarrow (\ell^2)^2$ is the identity. Then, the flow of the linearized system $\dot{\tilde{u}} = \tilde{\mathcal{L}}\tilde{u}$ preserves the energy norm on Y ,

$$\begin{aligned} \|\tilde{u}\|_Y^2 &:= \|u\|_E^2 + \|\mathbf{u}\|_M^2 \\ &= \sum_{j \in \mathbb{Z}} \left(v_1 (|u_{j+1,2} - u_{j,1}|^2 + |u_{j,1} - u_{j,2}|^2) + Mw_{1,1}|u_{j,1}|^2 + mw_{2,1}|u_{j,2}|^2 \right) \\ &\quad + \sum_{j \in \mathbb{Z}} (M|\mathbf{u}_{j,1}|^2 + m|\mathbf{u}_{j,2}|^2) \quad \text{for } \tilde{u} = \begin{pmatrix} u \\ \mathbf{u} \end{pmatrix}, \end{aligned}$$

i.e. its associated semi-group $e^{t\tilde{\mathcal{L}}}$ satisfies $\|e^{t\tilde{\mathcal{L}}}\|_{Y \rightarrow Y} = 1$, and from (21) it follows by Fourier transformation that the norms $\|\cdot\|_E$, $\|\cdot\|_M$ and $\|\cdot\|_{(\ell^2)^2}$, and hence also $\|\cdot\|_Y$ and $\|\cdot\|_{(\ell^2)^4}$, are equivalent: $\hat{\kappa}_1 \|u\|_{(\ell^2)^2} \leq \|u\|_M \leq \hat{\kappa}_2 \|u\|_{(\ell^2)^2}$, $\kappa_1 \|u\|_{(\ell^2)^2} \leq \|u\|_E \leq \kappa_2 \|u\|_{(\ell^2)^2}$, and $\check{\kappa}_1 \|\tilde{u}\|_{(\ell^2)^4} \leq \|\tilde{u}\|_Y \leq \check{\kappa}_2 \|\tilde{u}\|_{(\ell^2)^4}$, with $\hat{\kappa}_i, \kappa_i, \check{\kappa}_i > 0$ and $\|u\|_{(\ell^2)^2}^2 = \|u_1\|_{\ell^2}^2 + \|u_2\|_{\ell^2}^2$ for $u = (u_1, u_2)^T$, $\|\tilde{u}\|_{(\ell^2)^4}^2 = \|u\|_{(\ell^2)^2}^2 + \|\mathbf{u}\|_{(\ell^2)^2}^2$.

We consider the error $\varepsilon^\beta \tilde{R}_\varepsilon := \tilde{u} - \tilde{U}_\varepsilon^{A,2} = (u - U_\varepsilon^{A,2}, \dot{u} - \dot{U}_\varepsilon^{A,2})^T$ between an original solution u of (3) and the improved approximation $U_\varepsilon^{A,2}$ given by (13) with the $A_{1,n}$, $A_{2,\ell}$ determined by the formal derivation. Since for this $U_\varepsilon^{A,2}$ we have $\mathcal{L}U_\varepsilon^{A,2} + \mathcal{M}(U_\varepsilon^{A,2}) - \ddot{U}_\varepsilon^{A,2} = \text{res}(U_\varepsilon^{A,2})$, inserting \tilde{R}_ε into (22) we obtain the differential equation

$$\dot{\tilde{R}}_\varepsilon = \tilde{\mathcal{L}}\tilde{R}_\varepsilon + \varepsilon^{-\beta} \begin{pmatrix} 0 \\ \mathcal{M}(U_\varepsilon^{A,2} + \varepsilon^\beta R_\varepsilon) - \mathcal{M}(U_\varepsilon^{A,2}) + \text{res}(U_\varepsilon^{A,2}) \end{pmatrix}.$$

Taking the energy norm of its integral formulation, assuming $\|\tilde{R}_\varepsilon(0)\|_Y \leq d$, and applying Lemma 4.2 c), we get

$$(23) \quad \|\tilde{R}_\varepsilon(t)\|_Y \leq d + \varepsilon^{3/2-\beta} \tau_0 c_r + \varepsilon^{-\beta} \int_0^t \|\mathcal{M}(U_\varepsilon^{A,2} + \varepsilon^\beta R_\varepsilon) - \mathcal{M}(U_\varepsilon^{A,2})\|_M ds$$

for $\varepsilon \in (0, \varepsilon_0]$, $t \in [0, \tau_0/\varepsilon]$. From (3) and (2) we get by the mean value theorem

$$(24) \quad \|\mathcal{M}(u) - \mathcal{M}(\mathbf{u})\|_M \leq c_{\mathcal{M}} (\|u\|_{\ell^\infty} + \|\mathbf{u}\|_{\ell^\infty}) \|u - \mathbf{u}\|_M \quad \text{for } \|u\|_{\ell^\infty}, \|\mathbf{u}\|_{\ell^\infty} \leq c_0$$

with $\|u\|_{\ell^\infty} = \max\{\|u_1\|_{\ell^\infty}, \|u_2\|_{\ell^\infty}\}$ for $u = (u_1, u_2)^T \in (\ell^2)^2$, and $c_{\mathcal{M}}$ depending only on V_i, W_i and $c_0 > 0$.

We set $D := (d + \varepsilon_0^{3/2-\beta} \tau_0 c_r) e^{\tau_0 3c_{\mathcal{M}} c_A \hat{\kappa}_2 / \kappa_1}$ with c_A from Lemma 4.2 a) and $\varepsilon_0 > 0$ such that $\varepsilon_0^\beta D / \kappa_1 \leq \varepsilon_0 c_A \leq c_0/2$. Since $\|\tilde{R}_\varepsilon(0)\|_Y \leq d < D$ and $\|\tilde{R}_\varepsilon(t)\|_Y$ is continuous, there exists for every $\varepsilon \in (0, \varepsilon_0]$ a $t_D^\varepsilon > 0$, such that $\|\tilde{R}_\varepsilon(t)\|_Y \leq D$ for $t \in [0, t_D^\varepsilon]$. Then, for $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, \min\{\tau_0/\varepsilon, t_D^\varepsilon\}]$ (24) gives

$$\|\mathcal{M}(U_\varepsilon^{A,2} + \varepsilon^\beta R_\varepsilon) - \mathcal{M}(U_\varepsilon^{A,2})\|_M \leq \varepsilon^{\beta+1} (3c_{\mathcal{M}} c_A \hat{\kappa}_2 / \kappa_1) \|\tilde{R}_\varepsilon\|_Y.$$

Inserting this estimate into (23) and applying Gronwall's Lemma, we get

$$\|\tilde{R}_\varepsilon(t)\|_Y \leq \left(d + \varepsilon_0^{3/2-\beta} \tau_0 c_r \right) e^{\varepsilon t 3c_{\mathcal{M}} c_A \hat{\kappa}_2 / \kappa_1} \leq D \quad \text{for } \varepsilon \in (0, \varepsilon_0], t \in [0, \tau_0/\varepsilon].$$

Finally, with $d := \tilde{\kappa}_2 c + \varepsilon_0^{3/2-\beta} c_I$ and $C := (D + \varepsilon_0^{3/2-\beta} c_I)/\tilde{\kappa}_1$ we obtain from Lemma 4.2 b) and the equivalence of $\|\cdot\|_{(\ell^2)^4}$ and $\|\cdot\|_Y$ the assertion of the theorem. \square

Lemma 4.2. *For $U_\varepsilon^{A,2}$ given by (13) with $A_{1,n}, A_{2,\iota}$ as determined in Section 3 and initial data $A_{1,n}^{(i)}(0, \cdot) \in H^4(\mathbb{R}; \mathbb{C})$, there exist $\tau_0, \varepsilon_0, c_A, c_I, c_r > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon t \in [0, \tau_0]$*

$$a) \|U_\varepsilon^{A,2}\|_{\ell^\infty} \leq \varepsilon c_A, \quad b) \|\tilde{U}_\varepsilon^{A,2} - \tilde{U}_\varepsilon^{A,1}\|_Y \leq \varepsilon^{3/2} c_I, \quad c) \|\text{res}(U_\varepsilon^{A,2})\|_M \leq \varepsilon^{5/2} c_r.$$

Proof. Inserting into (12), (13) and $\text{res}(U_\varepsilon^{A,2})$ (see Appendix) $A_{1,n}^{(2)} = -\rho_n A_{1,n}^{(1)}$ (with $\rho_n = 0$ for $\vartheta_n = \pm\pi$, $\omega_n = \sqrt{c_1}$), $A_{2,n}^{(1)} \equiv 0$ [or $A_{1,n}^{(1)} \equiv A_{2,n}^{(2)} \equiv 0$ for $\vartheta_n = \pm\pi$, $\omega_n = \sqrt{c_2}$], and the $A_{2,n}^{(2)}$ [or $A_{2,n}^{(1)}$, respectively], $A_{2,\iota}$, $\iota \in I$, specified in Section 3, recalling $|\tilde{V}'_i(x)|, |\tilde{W}'_i(x)| = O(|x|^3)_{x \rightarrow 0}$ and the equivalence of $\|\cdot\|_E$, $\|\cdot\|_M$, $\|\cdot\|_{(\ell^2)^2}$, and using the corollary of Sobolev's embedding theorem (cf., e.g., [3, Lemma 3.1])

$$\|\varphi(\varepsilon(\cdot + \xi))\|_{\ell^2} \leq c\varepsilon^{-1/2} \|\varphi\|_{H^1(\mathbb{R}; \mathbb{C})}, \quad \xi : \mathbb{Z} \rightarrow [-1, 1], \quad \varepsilon \in (0, \varepsilon_0],$$

and $A, B \in C([0, \tau_0]; H^1(\mathbb{R}; \mathbb{C})) \Rightarrow AB \in C([0, \tau_0]; H^1(\mathbb{R}; \mathbb{C}))$, we obtain that the above estimates are satisfied, provided

$$\partial_\tau^p \partial_y^q A_{1,n}^{(1[2, \text{resp.}])} \in C([0, \tau_0]; H^1(\mathbb{R}; \mathbb{C})) \quad \text{for } |(p, q)| \leq 2, (p, q) = (3, 0), (2, 1).$$

Since the macroscopic equations for $A_{1,n}^{(1[2])}$, $n = 1, 2$, are semilinear autonomous transport systems with smooth nonlinearities, standard results of semigroup theory (cf., e.g., [6, Th. 6.1.7]) yield that for initial data $A_{1,n}^{(1[2])}(0, \cdot) \in H^m(\mathbb{R}; \mathbb{C})$, $m \geq 1$, there exist unique classical solutions with $\partial_\tau^p \partial_y^q A_{1,n}^{(1[2])} \in C([0, \tau_0]; H^1(\mathbb{R}; \mathbb{C}))$ for $|(p, q)| \leq m - 1$ up to some $\tau_0 \in (0, \infty]$. Hence, for $m = 4$ we obtain the statement of the lemma. \square

5. APPENDIX

For completeness we present here the $K_\iota = (K_\iota^{(1)}, K_\iota^{(2)})^T$, $\iota \in I$, and $\text{res}(U_\varepsilon^{A,2}) = (\text{res}_1(U_\varepsilon^{A,2}), \text{res}_2(U_\varepsilon^{A,2}))^T$ derived in Section 3, with $i = 1, 2$ (where $i + 1 = 1$ for $i = 2$) and with the upper sign corresponding to $i = 1$.

$$K_{(n,n)}^{(i)} := \pm v_{i,2} \left((A_{1,n}^{(i+1)})^2 (e^{\pm i 2 \vartheta_n} - 1) - 2 A_{1,n}^{(1)} A_{1,n}^{(2)} (e^{\pm i \vartheta_n} - 1) \right) - w_{i,2} (A_{1,n}^{(i)})^2,$$

$$K_{(1,2)}^{(i)} := \pm 2 v_{i,2} \left(A_{1,1}^{(i+1)} A_{1,2}^{(i+1)} (e^{\pm i(\vartheta_1 + \vartheta_2)} - 1) - A_{1,1}^{(i)} A_{1,2}^{(i+1)} (e^{\pm i \vartheta_2} - 1) \right. \\ \left. - A_{1,1}^{(i+1)} A_{1,2}^{(i)} (e^{\pm i \vartheta_1} - 1) \right) - 2 w_{i,2} A_{1,1}^{(i)} A_{1,2}^{(i)},$$

$$K_{(1,-2)}^{(i)} := \pm 2 v_{i,2} \left(A_{1,1}^{(i+1)} \bar{A}_{1,2}^{(i+1)} (e^{\pm i(\vartheta_1 - \vartheta_2)} - 1) - A_{1,1}^{(i)} \bar{A}_{1,2}^{(i+1)} (e^{\mp i \vartheta_2} - 1) \right. \\ \left. - A_{1,1}^{(i+1)} \bar{A}_{1,2}^{(i)} (e^{\pm i \vartheta_1} - 1) \right) - 2 w_{i,2} A_{1,1}^{(i)} \bar{A}_{1,2}^{(i)},$$

$$K_{(1,-1)}^{(i)} := \sum_{n=1}^2 \left(\mp 2 v_{i,2} \bar{A}_{1,n}^{(i)} A_{1,n}^{(i+1)} (e^{\pm i \vartheta_n} - 1) - w_{i,2} |A_{1,n}^{(i)}|^2 \right);$$

$$\text{res}_i(U_\varepsilon^{A,2}) := \varepsilon^3 \left(-T_i \pm v_{i,1} F_i \pm 2 v_{i,2} (D_i E_i - (A_1 - A_2)(B_1 - B_2)) - 2 w_{i,2} A_i B_i \right) \\ + \varepsilon^4 \left(-S_i \pm v_{i,2} (E_i^2 + 2 D_i F_i - (B_1 - B_2)^2) - w_{i,2} B_i^2 \right) \pm \varepsilon^5 v_{i,2} E_i F_i \pm \varepsilon^6 v_{i,2} F_i^2$$

$$\begin{aligned}
& \pm \tilde{V}'_i(\varepsilon D_i + \varepsilon^2 E_i + \varepsilon^3 F_i) \mp \tilde{V}'_i(\varepsilon(A_1 - A_2) + \varepsilon^2(B_1 - B_2)) - \tilde{W}'_i(\varepsilon A_i + \varepsilon^2 B_i), \\
T_i &:= \sum_{n=1}^2 \left((\partial_\tau^2 A_{1,n}^{(i)} + \partial_\tau A_{2,n}^{(i)} 2i\omega_n) \mathbf{E}_n + \partial_\tau A_{2,(n,n)}^{(i)} 4i\omega_n \mathbf{E}_n^2 \right) \\
& \quad + \partial_\tau A_{2,(1,2)}^{(i)} 2i(\omega_1 + \omega_2) \mathbf{E}_1 \mathbf{E}_2 + \partial_\tau A_{2,(1,-2)}^{(i)} 2i(\omega_1 - \omega_2) \mathbf{E}_1 \mathbf{E}_{-2} + \text{c.c.}, \\
S_i &:= \sum_{n=1}^2 \left(\partial_\tau^2 A_{2,n}^{(i)} \mathbf{E}_n + \partial_\tau^2 A_{2,(n,n)}^{(i)} \mathbf{E}_n^2 \right) + \partial_\tau^2 A_{2,(1,2)}^{(i)} \mathbf{E}_1 \mathbf{E}_2 + \partial_\tau^2 A_{2,(1,-2)}^{(i)} \mathbf{E}_1 \mathbf{E}_{-2} \\
& \quad + \frac{1}{2} \partial_\tau^2 A_{2,(1,-1)}^{(i)} + \text{c.c.}, \\
A_i &:= \sum_{n=1}^2 A_{1,n}^{(i)} \mathbf{E}_n + \text{c.c.}, \quad D_i := \pm \sum_{n=1}^2 (A_{1,n}^{(i+1)} e^{\pm i\vartheta_n} - A_{1,n}^{(i)}) \mathbf{E}_n + \text{c.c.}, \\
B_i &:= \sum_{n=1}^2 \left(A_{2,n}^{(i)} \mathbf{E}_n + A_{2,(n,n)}^{(i)} \mathbf{E}_n^2 \right) + A_{2,(1,2)}^{(i)} \mathbf{E}_1 \mathbf{E}_2 + A_{2,(1,-2)}^{(i)} \mathbf{E}_1 \mathbf{E}_{-2} + \frac{1}{2} A_{2,(1,-1)}^{(i)} \\
& \quad + \text{c.c.}, \\
E_i &:= \sum_{n=1}^2 \left((\partial_y A_{1,n}^{(i+1)} e^{\pm i\vartheta_n} \pm A_{2,n}^{(i+1)} e^{\pm i\vartheta_n} \mp A_{2,n}^{(i)}) \mathbf{E}_n \pm (A_{2,(n,n)}^{(i+1)} e^{\pm i2\vartheta_n} - A_{2,(n,n)}^{(i)}) \mathbf{E}_n^2 \right) \\
& \quad \pm (A_{2,(1,2)}^{(i+1)} e^{\pm i(\vartheta_1 + \vartheta_2)} - A_{2,(1,2)}^{(i)}) \mathbf{E}_1 \mathbf{E}_2 \pm (A_{2,(1,-2)}^{(i+1)} e^{\pm i(\vartheta_1 - \vartheta_2)} - A_{2,(1,-2)}^{(i)}) \mathbf{E}_1 \mathbf{E}_{-2} \\
& \quad + \frac{1}{2} (A_{2,(1,-1)}^{(2)} - A_{2,(1,-1)}^{(1)}) + \text{c.c.}, \\
F_i &:= \sum_{n=1}^2 \left(\left(\pm \frac{1}{2} \partial_y^2 A_{1,n,\xi_1\pm}^{(i+1)} + \partial_y A_{2,n,\xi_2\pm}^{(i+1)} \right) e^{\pm i\vartheta_n} \mathbf{E}_n + \partial_y A_{2,(n,n),\xi_2\pm}^{(i+1)} e^{\pm i2\vartheta_n} \mathbf{E}_n^2 \right) \\
& \quad + \partial_y A_{2,(1,2),\xi_2\pm}^{(i+1)} e^{\pm i(\vartheta_1 + \vartheta_2)} \mathbf{E}_1 \mathbf{E}_2 + \partial_y A_{2,(1,-2),\xi_2\pm}^{(i+1)} e^{\pm i(\vartheta_1 - \vartheta_2)} \mathbf{E}_1 \mathbf{E}_{-2} \\
& \quad + \frac{1}{2} \partial_y A_{2,(1,-1),\xi_2\pm}^{(i+1)} + \text{c.c.}
\end{aligned}$$

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